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Some relationship between degenerate (p, q)-Euler polynomials and other polynomials

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Abstract. We construct new degenerate (p, q)-Euler polynomials and find some properties and identities of these polynomials. It can be seen that the degenerate (p, q)-Euler polynomials which is related to Euler polynomials have various relation with other polynomials.

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1. Introduction

Before clarifying the objectives of this paper, we first introduce the necessary basic concepts. We identify several definitions and properties and present the goals of this paper based on them.

For any $n \in \mathbb{C}$, the (p, q)-number [3, 11, 12] is defined by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}.$$

Definition 1.1 [3], [11]. Let z be any complex numbers with $|z| < 1$. The two forms of (p, q)-exponential functions are defined by

$$e_{p,q}(z) = \sum_{n=0}^{\infty} p^{\binom{n}{2}} \frac{z^n}{[n]_{p,q}!},$$

$$E_{p,q}(z) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{z^n}{[n]_{p,q}!}.$$

Definition 1.2 [11]. Let $n \geq k$. (p,q)-Gauss Binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[n-k]_{p,q}![k]_{p,q}!},$$

where $[n]_{p,q}! = [n]_{p,q}[n-1]_{p,q}\cdots[1]_{p,q}$.

Definition 1.3 [12]. (p,q)-Euler numbers $\mathcal{E}_{n,p,q}$ and polynomials $\mathcal{E}_{n,p,q}(x)$ are defined by

$$\sum_{n=0}^{\infty} \mathcal{E}_{n,p,q} \frac{t^n}{[n]_{p,q}!} = \frac{2}{e_{p,q}(t)+1},$$
$$\sum_{n=0}^{\infty} \mathcal{E}_{n,p,q}(x) \frac{t^n}{[n]_{p,q}!} = \frac{2}{e_{p,q}(t)+1} e_{p,q}(tx).$$

Consider $p = 1$ in Definition 1.5. Then, we note

$$\sum_{n=0}^{\infty} \mathcal{E}_{n,q} \frac{t^n}{[n]_{q}!} = \frac{2}{e_{q}(t)+1},$$
$$\sum_{n=0}^{\infty} \mathcal{E}_{n,q}(x) \frac{t^n}{[n]_{q}!} = \frac{2}{e_{q}(t)+1} e_{q}(tx),$$

where $\mathcal{E}_{n,q}$ is the q-Euler number and $\mathcal{E}_{n,q}(x)$ is the q-Euler polynomial, see [6,8, 9].

Definition 1.4 [2], [10]. Let $f : \mathbf{T}_{q,h} \to \mathbb{R}$ be any function. Then, the delta (q,h)-derivative of f $D_{q,h}(f)$ is defined by

$$D_{q,h}f(x) := \frac{f(qx+h) - f(x)}{(q-1)x+h}.$$

Definition 1.5 [1]. The generalized quantum binomial $(x - x_0)^n_{q,h}$ is defined by

$$(x - x_0)^n_{q,h} := \begin{cases} 1, & \text{if } n = 0, \\ \prod_{i=1}^{n} \left(x - (q^{i-1}x_0 + [i-1]_q h)\right), & \text{if } n > 0, \end{cases}$$

where $x_0 \in \mathbb{R}$.

The generalized quantum binomials reduces to q-binomial $(x-x_0)_q^n$ as $h \to 0$ and to h-binomial $(x-x_0)_h^n$ when $q \to 1$. Also, we note $\lim_{(q,h)\to(1,0)}(x-x_0)_{q,h}^n = (x-x_0)^n$.

Definition 1.6 [2]. The generalized quantum exponential function $\exp_{q,h}(\alpha x)$ is defined as

$$\exp_{q,h}(\alpha x) := \sum_{i=0}^{\infty} \frac{\alpha^i (x-0)_{q,h}^i}{[i]_q!},$$

where α is arbitrary nonzero constant.

Based on the above concepts, many mathematicians have studied q-special functions, q-differential equations, q-calculus, and so on, see [4-9].

The purpose of this paper is to define new type of Euler polynomials and to find various relations related to these polynomials.

2. Some properties of degenerate (p, q)-Euler polynomials

We introduce the following degenerate (p, q) exponential function as

$$\begin{aligned} e_{p,q,h}(x:t) &:= \sum_{n=0}^{\infty} (x)_{p,q,h}^n \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} (x-0)_{p,q,h}^n \frac{t^n}{n!}. \end{aligned}$$

For example, substituting $x = 1$ in the above equation, we have

$$e_{p,q,h}(1:t) = \sum_{n=0}^{\infty} (1)_{p,q,h}^n \frac{t^n}{n!},$$

where $(1)_{p,q,h}^n = (1-0)_{p,q,h}^n = 1(1-h)\cdots(1-[n-1]_q h)$.

Definition 2.1. Let $|p/q| < 1$ and h be a non-negative integer. Then, we define the degenerate (p,q)-Euler polynomials $\mathscr{E}_{n,p,q}(x:h)$ as

$$\begin{aligned}&\sum_{n=0}^{\infty}\mathscr{E}_{n,p,q}(x:h)\frac{t^n}{[n]_{p,q}!}\\&=\frac{2}{e_{p,q,h}(1:t)+1}e_{p,q,h}(x:t).\end{aligned}$$

For $x=0$ in Definition 2.1, we have

$$\sum_{n=0}^{\infty}\mathscr{E}_{n,p,q}(h)\frac{t^n}{[n]_q!}=\frac{2}{e_{p,q,h}(1:t)+1},$$

and we call $\mathscr{E}_{n,p,q}(h)$ is the degenerate (p,q)-Euler numbers. From Definition 2.1, we can see several relationships for Euler polynomials.

Setting $h\to 0$ in Definition 2.1, we find the (p,q)-Euler numbers $\mathcal{E}_{n,p,q}$ and polynomials $\mathcal{E}_{n,p,q}(x)$ as

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{E}_{n,p,q}\frac{t^n}{[n]_{p,q}!}&=\frac{2}{e_{p,q}(t)+1},\\\sum_{n=0}^{\infty}\mathcal{E}_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!}&=\frac{2}{e_{p,q}(t)+1}e_{p,q}(tx).\end{aligned}$$

Let $h\to 0,\ q\to 1$ and $p=1$ in Definition 2.1. Then, we have the Euler numbers $\mathcal{E}_n$ and polynomials $\mathcal{E}_n(x)$ as

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{E}_n\frac{t^n}{n!}&=\frac{2}{e^t+1},\\\sum_{n=0}^{\infty}\mathcal{E}_n(x)\frac{t^n}{n!}&=\frac{2}{e^t+1}e^tx.\end{aligned}$$

In addition, for $p=1$ and $q\to 1$ in Definition 2.1, we see the degenerate Euler numbers $\mathscr{E}_n(h)$ and polynomials $\mathscr{E}_n(x:h)$ as follows:

$$\begin{aligned}\sum_{n=0}^{\infty}\mathscr{E}_n(h)\frac{t^n}{n!}&=\frac{2}{(1+ht)^{\frac{1}{h}}+1},\\\sum_{n=0}^{\infty}\mathscr{E}_n(x:h)\frac{t^n}{n!}&=\frac{2}{(1+ht)^{\frac{1}{h}}+1}(1+ht)^{\frac{x}{h}},\end{aligned}$$

where $\mathscr{E}_n(h) = \mathscr{E}_n(0:h)$.

Theorem 2.2. *Let k be a non-negative integer with $|p/q| < 1$. Then we have*

(i)

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} (1)_{p,q,h}^{k} \mathscr{E}_{n-k,p,q,h}(h) + \mathscr{E}_{n,p,q}(h) = \begin{cases} 2, & \text{if } n = 0, \\ 0, & \text{if } n \geq 1. \end{cases}$$

(ii)

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} (1)_{p,q,h}^{k} \mathscr{E}_{n-k,p,q,h}(x:h) + \mathscr{E}_{n,p,q}(x:h) = 2(x)_{p,q,h}^{n}.$$

Proof.

(i) To find the required result, we suppose $e_{p,q,h}(1:t) \neq -1$. Then, we obtain

$$\begin{aligned} &\sum_{n=0}^{\infty} \mathscr{E}_{n,p,q}(h) \frac{t^n}{[n]_{p,q}!} \left(e_{p,q,h}(1:t) + 1\right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} (1)_{p,q,h}^{k} \mathscr{E}_{n-k,p,q,h}(h) + \mathscr{E}_{n,p,q}(h) \right) \frac{t^n}{[n]_{p,q}!} \\ &= 2. \end{aligned} \tag{1}$$

Comparing the coefficients of both-sides in Equation (1), we finish the proof of Theorem 2.2. (i).

(ii) From the generating function of the degenerate (p,q)-Euler polynomials with the same condition of (i), we have

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}(1)_{p,q,h}^{k}\mathscr{E}_{n-k,p,q,h}(x:h)+\mathscr{E}_{n,p,q}(x:h)\right)\frac{t^n}{[n]_{p,q}!}$$
$$=2\sum_{n=0}^{\infty}(x)_{p,q,h}^{n}\frac{t^n}{[n]_{p,q}!}. \tag{2}$$

From (2), we complete the proof of Theorem 2.2. (ii). □

Theorem 2.3. *For $|p/q|<1$, we derive*

$$\mathscr{E}_{n,p,q}(x:h)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}(x)_{p,q,h}^{n-k}\mathscr{E}_{k,p,q}(h).$$

Proof. To find a relation between $\mathscr{E}_{n,p,q}(x:h)$ and $\mathscr{E}_{n,p,q}(h)$, we can be transformed as

$$\begin{aligned}\sum_{n=0}^{\infty}\mathscr{E}_{n,p,q}(x:h)\frac{t^n}{[n]_{p,q}!}&=\frac{2}{e_{p,q,h}(1:t)+1}e_{p,q,h}(x:t)\\&=\sum_{n=0}^{\infty}\mathscr{E}_{n,p,q}(h)\frac{t^n}{[n]_{p,q}!}\sum_{n=0}^{\infty}(x)_{p,q,h}^{n}\frac{t^n}{[n]_{p,q}!}\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}(x)_{p,q,h}^{n-k}\mathscr{E}_{k,p,q}(h)\right)\frac{t^n}{[n]_{p,q}!}.\end{aligned} \tag{3}$$

Using the coefficient comparison method, we find the required result. □

Corollary 2.4. *From Theorem 2.1, the following hold:*

(i) *Setting $p=1$, we have*

$$\mathscr{E}_{n,q}(x:h)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{q}(x)_{q,h}^{n-k}\mathscr{E}_{k,q}(h),$$

where $\begin{bmatrix}n\\k\end{bmatrix}_{q}$ is the q-binomial coefficient and

$$(x)_{q,h}^{n}=x(x-h)(x-[2]_qh)\cdots(x-[n-1]_qh).$$

(ii) *Setting $p = 1$ and $q \to 1$, we have*

$$\mathscr{E}_n(x : h) = \sum_{k=0}^{n} \binom{n}{k} (x)_h^{n-k} \mathscr{E}_k(h),$$

where

$$(x)_h^n = x(x-h)(x-2h)\cdots(x-(n-1)h).$$

(iii) *Setting $p = 1$ and $h \to 0$, we have*

$$\mathcal{E}_{n,q}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^{n-k} \mathcal{E}_{k,q},$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the q-binomial coefficient.

Theorem 2.5. *Let n, k be any non-negative integers. Then, we have*

(i)

$$\begin{aligned} &2\mathscr{E}_{n,p,q}(x : h) \\ &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} (\mathcal{E}_{k,p,q}(1) + \mathcal{E}_{k,p,q})\, \mathscr{E}_{n-k,p,q,h}(x : h), \end{aligned}$$

where $\mathcal{E}_{n,p,q}$ is the (p,q)-Euler numbers and $\mathcal{E}_{n,p,q}(x)$ is the (p,q)-Euler polynomials.

(ii)

$$\begin{aligned} &[n]_{p,q}\mathscr{E}_{n-1,p,q}(x : h) \\ &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} (B_{k,p,q}(1) - B_{k,p,q})\, \mathscr{E}_{n-k,p,q,h}(x : h), \end{aligned}$$

where $B_{n,p,q}$ is the (p,q)-Bernoulli numbers and $B_{n,p,q}(x)$ is the (p,q)-Bernoulli polynomials.

(iii)

$$2[n]_{p,q}\mathscr{E}_{n-1,p,q}(x:h) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}\left(G_{k,p,q}(1)+G_{k,p,q}\right)\mathscr{E}_{n-k,p,q,h}(x:h),$$

where $G_{n,p,q}$ is the (p,q)-Genocchi numbers and $G_{n,p,q}(x)$ is the (p,q)-Genocchi polynomials.

Proof. To find some relation of (p,q)-Euler, Bernoulli, and Genocchi polynomials, we consider that:

(i) (a relation of the degenerate (p,q)-Euler polynomials and (p,q)-Euler polynomials)

$$\begin{aligned}\sum_{n=0}^{\infty}&\mathscr{E}_{n,p,q}(x:h)\frac{t^n}{[n]_{p,q}!}\\ &=\left(\frac{1}{e_{p,q}(t)+1}e_{p,q}(t)+\frac{1}{e_{p,q}(t)+1}\right)\frac{2}{e_{q,h}(1:t)+1}e_{q,h}(x:t)\\ &=\frac{1}{2}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}\left(\mathcal{E}_{k,p,q}(1)+\mathcal{E}_{k,p,q}\right)\mathscr{E}_{n-k,p,q,h}(x:h)\right)\\ &\frac{t^n}{[n]_{p,q}!}.\end{aligned} \tag{4}$$

(ii) (a relation of the degenerate (p,q)-Euler polynomials and (p,q)-Bernoulli polynomials)

$$\begin{aligned}\sum_{n=0}^{\infty}&\mathscr{E}_{n,p,q}(x:h)\frac{t^n}{[n]_{p,q}!}\\ &=\left(\frac{1}{e_{p,q}(t)-1}e_{p,q}(t)-\frac{1}{e_{p,q}(t)-1}\right)\frac{2}{e_{q,h}(1:t)+1}e_{q,h}(x:t)\\ &=\frac{1}{t}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}\left(B_{k,p,q}(1)-B_{k,p,q}\right)\mathscr{E}_{n-k,p,q,h}(x:h)\right)\\ &\frac{t^n}{[n]_{p,q}!}.\end{aligned} \tag{5}$$

(iii) (a relation of the degenerate (p,q)-Euler polynomials and (p,q)-Genocchi polynomials)

$$\sum_{n=0}^{\infty}\mathscr{E}_{n,p,q}(x:h)\frac{t^n}{[n]_{p,q}!}$$
$$=\left(\frac{1}{e_{p,q}(t)+1}e_{p,q}(t)+\frac{1}{e_{p,q}(t)+1}\right)\frac{2}{e_{q,h}(1:t)+1}e_{q,h}(x:t)$$
$$=\frac{1}{2t}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\left(G_{k,p,q}(1)+G_{k,p,q}\right)\mathscr{E}_{n-k,p,q,h}(x:h)\right)$$
$$\frac{t^n}{[n]_{p,q}!}. \tag{6}$$

From (4), (5), and (6), we get the result. □

Corollary 2.6. *Setting $p=1$ in Theorem 2.5, we have*

(i)

$$2\mathscr{E}_{n,q}(x:h)$$
$$=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{q}\left(\mathcal{E}_{k,q}(1)+\mathcal{E}_{k,q}\right)\mathscr{E}_{n-k,q,h}(x:h),$$

where $\mathcal{E}_{n,q}$ is the q-Euler numbers and $\mathcal{E}_{n,q}(x)$ is the q-Euler polynomials.

(ii)

$$[n]_q\mathscr{E}_{n-1,q}(x:h)$$
$$=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{q}\left(B_{k,q}(1)-B_{k,q}\right)\mathscr{E}_{n-k,q,h}(x:h),$$

where $B_{n,q}$ is the q-Bernoulli numbers and $B_{n,q}(x)$ is the q-Bernoulli polynomials.

(iii)

$$2[n]_q\mathscr{E}_{n-1,q}(x:h)$$
$$=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{q}\left(G_{k,q}(1)+G_{k,q}\right)\mathscr{E}_{n-k,q,h}(x:h),$$

where $G_{n,q}$ is the q-Genocchi numbers and $G_{n,q}(x)$ is the q-Genocchi polynomials.

Theorem 2.7. *For $|q| < 1$, we have*

$$2\mathscr{E}_{n,p,q}(x:h) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \frac{\begin{bmatrix} n \\ k \end{bmatrix}_q [n-k]_{p^{-1}q}! p^{\binom{n-k}{2}} (\mathcal{E}_{k,q}(1) + \mathcal{E}_{k,q})}{[n]_q[n-1]_q \cdots [n-k+1]_q} \mathscr{E}_{n-k,p,q}(x:h),$$

where $\mathcal{E}_{n,q}(x)$ is the q-Euler polynomials and $\mathcal{E}_{n,q}$ is the q-Euler numbers.

Proof. To derive a relation of $\mathscr{E}_{n,p,q}(x : h)$ and $\mathcal{E}_{k,q}(x)$, we find as the follows:

$$\begin{aligned} &\sum_{n=0}^{\infty} \mathscr{E}_{n,p,q}(x:h) \frac{t^n}{[n]_q!} \\ &= \left(\frac{1}{e_q(t)+1} e_q(t) + \frac{1}{e_q(t)+1} \right) \frac{2}{e_{q,h}(1:t)+1} e_{q,h}(x:t) \\ &= \frac{1}{2} \sum_{n=0}^{\infty} (\mathcal{E}_{k,q}(1) + \mathcal{E}_{k,q}) \frac{t^n}{[n]_q!} \\ &\sum_{n=0}^{\infty} \mathscr{E}_{n-k,p,q,h}(x:h) \frac{t^n}{[n]_{p,q}!}. \end{aligned} \tag{7}$$

Here, we obtain a relation $[n]_{p,q}! = p^{\binom{n}{2}}[n]_{p^{-1}q!}$. Using this relation in (7), we have the result of Theorem 2.7. □

3. Conclusion

In this paper, we constructed the degenerate (p, q)-Euler polynomials and found some properties of these polynomials. We also derived several relationship between degenerate (p, q)-Euler polynomials and other polynomials.

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